Criteria for existence and stability of soliton solutions of the cubic-quintic nonlinear Schrödinger equation

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A subset of the soliton solutions of the cubic-quintic nonlinear Schrödinger equation (NLSE) is presented in analytical form. General criteria for existence are expressed in terms of the parameters of the NLSE. The normalized momentum entering the stability criterion is evaluated explicitly.

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I. INTRODUCTION

Many physical problems can be described by the nonlinear Schrödinger equation (NLSE) [1,2]. One of the main features of the NLSE is the existence of a special class of localized solutions, solitons, which are robust against perturbations and demonstrate a particlelike behavior. In particular, spatial solitons have attracted some interest in optics because of their possible use in optical communication [3].

In all applications of solitons, the key problem is to find and evaluate criteria of the existence and stability expressed in terms of the parameters of the NLSE.

Considering solitons over nonvanishing backgrounds, the stability of the soliton is determined by the relation between the soliton speed (traveling in a motionless background with a nonvanishing intensity J_∞) and the parameters of the NLSE. In this respect, the following analysis is based on and/or related to the results of Refs. [4–7]. Accordingly, solitons are stable [7] if $\partial P/\partial v<0$, where the renormalized momentum $P=i/2\int_{-\infty}^{+\infty}(\Psi_z^*\Psi-\Psi_z\Psi^*)(1-J_\infty/|\Psi|^2)dz$ for soliton solutions of the NLSE is given by [5,6]

$$P=v\int_{-\infty}^{+\infty}\frac{[J(z)-J_\infty]^2}{J(z)}dz, \tag{1}$$

and v and $J=|\Psi(z)|^2$ denote the velocity and the intensity of the soliton, respectively. In the spatial case, v defines the steering angle α of the soliton, $v=\tan\alpha$. If $\partial P/\partial v>0$, the soliton is unstable [4,7,8].

It was shown in a previous paper [9] (herein referred to as S) that a certain subset of soliton solutions of the NLSE exists and can be represented in closed form. Conditions for this existence were given by the phase diagram conditions (PDC) geometrically. In this paper we evaluate the PDC algebraically, leading to explicit representations of the intensity $J(z)$, the phase $g(z)$, the normalized momentum P, and analytical criteria for existence (Secs. II, III, and IV). Section V contains a numerical example and summarizes the theoretical discussions.

II. INTENSITY AND PHASE OF SOLITON SOLUTIONS

Graphs depicted in Figs. 5(a)–5(c) of S represent soliton solutions

$$\Psi(z,x)=\sqrt{J(z)}e^{ig(z)}e^{-i\lambda x}, \tag{2}$$

of the NLSE,

$$i\Psi_x+\Psi_{zz}=a_1\Psi|\Psi|^2+a_2\Psi|\Psi|^4,$$
$$a_i\in\mathbb{R},\quad \{a_1,a_2\}\neq\{0,0\}, \tag{3}$$

where [cf. Eq. (S10)]

$$R(J)=4\left(\frac{a_2}{3}J^4+\frac{a_1}{2}J^3-\lambda J^2+kJ-C^2\right)=(J_z)^2. \tag{4}$$

The diagrams can be specified algebraically. Since $\Delta=0$ and $g_2\geqslant 0$, $g_3\leqslant 0$ [cf. Eqs. (S12), (S13), and (S18)] are necessary conditions for the existence of solitons [9], Weierstrass's function in Eq. (S14) degenerates [10], leading to

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$$J(z)=\begin{cases} J_\infty-\dfrac{3j_1R'}{4\left(j_1-\dfrac{R''}{24}\right)\left[3j_1+\left(j_1-\dfrac{R''}{24}\right)\sinh^2(\sqrt{3j_1}z)\right]}\Bigg|_{J=J_0}, & j_1>0,\ g_3<0\\ J_\infty+\dfrac{6R'}{R''\left(1-\dfrac{R''}{24}z^2\right)}\Bigg|_{J=J_0}, & j_1=0,\ g_3=0, \end{cases} \tag{5}$$

with $j_1=\sqrt[3]{-g_3/8}>0$, $g_3=C^2(a_1^2+\frac{32}{9}a_2\lambda)-\frac{2}{3}k(2a_2k+a_1\lambda)+\frac{8}{27}\lambda^3$, and

$$J_\infty=J_0+\frac{R'}{4\left(j_1-\dfrac{R''}{24}\right)}\Bigg|_{J=J_0}. \tag{6}$$

J_0 is a simple positive root of $R(J)$. J_∞ denotes the multiple positive root of $R(J)$ (double if $j_1>0$, triple, if $j_1=0$). J_∞ is equal to the background intensity in Eq. (1). The prime in Eq. (5) indicates differentiation with respect to J. Equation (5) completely represents the intensity $J(z)$ of the soliton solutions ($g_3<0$ for nonalgebraic, $g_3=0$ for algebraic solitons) if the simple positive root J_0 of $R(J)$ can be expressed in terms of the coefficients of $R(J)$. As will be seen below, this is possible.

The phase $g(z)$ is related to $J(z)$ by [cf. Eq. (S8)] $J(z)\times(dg/dz)=C$. Introducing the asymptotic wave number $q=\lim_{|z|\to\infty}(dg/dz)$ of the background plane wave, the integration constant C can be written as $C=qJ_\infty$. Thus the phase is given by (cf. Ref. [5])

$$g(z)=J_\infty q\int\frac{dz}{J(z)}, \tag{7}$$

with $J(z)$ according to Eq. (6).

With respect to a stability analysis, it should be noted that the solution $J(z)$ according to Eq. (5) refers to a situation in which the background medium and the solitons have the same speed, so that the stability criterion cannot be applied directly because it refers to a background at rest supporting a soliton that travels with speed v relative to the background. Applying the transformation

$$\tilde{\Psi}(z',x')=\Psi(z,x)e^{-iqz},\quad z'=z+qx,\quad x'=x, \tag{8}$$

the background becomes quiescent while the soliton moves with speed $v=-q$, so that, according to Eqs. (5) and (7), the intensity $J(z)$ and the phase $g(z)$ depend nontrivially on the speed q. Obviously, the key problem of an existence analysis is to identify the simple positive root J_0 of the fourth-order polynomial $R(J)$ [cf. Figs. 5(a)–5(c) in S].

III. NONALGEBRAIC SOLITONS

First, in Eq. (5), the case of nonalgebraic solitons is considered ($j_1>0$, $g_3<0$). Since J_∞ is a double root, the integration constant k is related to J_∞ by

$$k=-\frac{4}{3}a_2J_\infty^3-\frac{3}{2}a_1J_\infty^2+2\lambda J_\infty. \tag{9}$$

Hence $R(J_\infty)$ can be written as

$$R(J_\infty)=-4J_\infty^2[J_\infty(a_1+a_2J_\infty)-\lambda+q^2], \tag{10}$$

leading to the double roots

$$J_{\infty\pm}=\frac{-a_1\pm\sqrt{a_1^2+4a_2(\lambda-q^2)}}{2a_2}. \tag{11}$$

The simple roots, associated to $J_{\infty\pm}$, are, respectively,

$$J_{0\pm}^{+}=\frac{-a_1-2\sqrt{a_1^2+4a_2(\lambda-q^2)}\pm\sqrt{48a_2q^2+[a_1+2\sqrt{a_1^2+4a_2(\lambda-q^2)}]^2}}{4a_2} \tag{12a}$$

and

$$J_{0\pm}^{-}=\frac{-a_1+2\sqrt{a_1^2+4a_2(\lambda-q^2)}\pm\sqrt{48a_2q^2+[a_1+2\sqrt{a_1^2+4a_2(\lambda-q^2)}]^2}}{4a_2}. \tag{12b}$$

The phase diagrams according to Figs. 5(a) and 5(b) of S can be specified by using Eqs. (11) and (12). If $a_2>0$ [Fig. 5(a) of S], there must be three and only three real positive roots of $R(J)=0$. It can be shown that the double root $J_{\infty-}$ must be disregarded since only $J_{\infty+}$ can be the largest real root of the associated polynominal R with k given by Eq. (9). For this case, the necessary and sufficient conditions of existence are

$$a_1^2+4a_2(\lambda-q^2)\geqslant 0, \tag{13a}$$

$$2q^2-\lambda<\frac{a_1^2-a_1\sqrt{a_1^2+4a_2(\lambda-q^2)}}{4a_2}, \tag{13b}$$

$$\sqrt{a_1^2+4a_2(\lambda-q^2)}+\frac{a_1}{2}>0, \tag{13c}$$

being the algebraic representation of the phase diagram in Fig. 5(a) of S associated to a dark soliton.

In the case of solitons according to Fig. 5(b) of S, four changes of sign in the sequence $\{a_2,a_1,-\lambda,k,-q^2J_\infty^2\}$ are necessary, leading to $a_2<0$, $a_1>0$, and hence it follows that $J_{\infty-}$ is either the greatest root of R or the associated simple roots are complex. Thus, soliton solutions do not exist if $J_{\infty-}$ is chosen in Eq. (5), so that only $J_{\infty+}\equiv J_\infty$ must be considered. Obviously, $J_{0+}^+<J_\infty<J_{0-}^+$ must hold, which is equivalent to

$$48a_2q^2+[a_1+2\sqrt{a_1^2+4a_2(\lambda-q^2)}]^2>0, \tag{14a}$$

$$2q^2-\lambda<\frac{a_1^2-a_1\sqrt{a_1^2+4a_2(\lambda-q^2)}}{4a_2}, \tag{14b}$$

subject to condition (13a). Conditions (14) represent the phase diagram in Fig. 5(b) of S. Thus, the intensity $J(z)$ and hence the phase $g(z)$ and the normalized momentum P of nonalgebraic solitons can be evaluated according to Eq. (5a) subject to conditions (13) and (14). If $a_2>0$, J_{0+}^+ must be chosen in Eq. (5) yielding a dark soliton. If $a_2<0$ holds, a bright and a dark soliton is represented by choosing J_{0-}^+ and J_{0+}^+ in Eq. (5), respectively. Evaluation yields, for bright and dark solitons, respectively,

$$g_\pm=q\left(z+\frac{b\tanh^{-1}\left(\sqrt{\dfrac{b-J_\infty\left(2j_1+\dfrac{R''}{24}\right)}{3j_1J_\infty-b}}\tanh\sqrt{3j_1}z\right)}{\sqrt{3j_1(3j_1I_\infty-b)\left[b-J_\infty\left(2j_1+\dfrac{R''}{24}\right)\right]}}\right)\Bigg|_{J=J_{0\pm}}, \tag{15}$$

$$P_\pm=\frac{qb}{\sqrt{3j_1}}\left(\frac{\tanh^{-1}\sqrt{\dfrac{2j_1+\dfrac{R''}{24}}{3j_1}}}{\sqrt{3j_1\left(2j_1+\dfrac{R''}{24}\right)}}-\frac{J_\infty\tanh^{-1}\sqrt{\dfrac{J_\infty\left(2j_1+\dfrac{R''}{24}\right)-b}{3j_1J_\infty-b}}}{\sqrt{(3j_1J_\infty-b)\left[J_\infty\left(2j_1+\dfrac{R''}{24}\right)-b\right]}}\right)\Bigg|_{J=J_{0\pm}}, \tag{16}$$

with $b=3j_1R'/4[j_1-(R''/24)]$, and

$$R(J)=4\left\{\frac{a_2}{3}J^4+\frac{a_2}{2}J^3-\lambda J^2+\left(2J_\infty\lambda-\frac{3}{2}a_1J_\infty^2-\frac{4a_2}{3}J_\infty^3\right)J-J_\infty^2q^2\right\}. \tag{17}$$

Numerical evaluation of $J(z)$, $g(z)$, P subject to the existence criteria (13) or (14) is straightforward, as illustrated in Sec. V.

IV. ALGEBRAIC SOLITONS

Turning to algebraic solitons [cf. Figs. 5(c) and 5(d) of S], $R(J)$ has a triple root $\tilde{J}_\infty$ so that, in addition to Eq. (5),

$$\lambda=2a_2\tilde{J}_\infty^2+\frac{3}{2}a_1\tilde{J}_\infty \tag{18}$$

must hold. Thus, the triple root is given by

$$\tilde{J}_{\infty\pm}=\frac{-a_1\pm\sqrt{a_1^2+16a_2q^2}}{4a_2}, \tag{19}$$

leading to $k_\pm$, $\lambda_\pm$ according to Eqs. (9) and (18).

The associated simple roots are

$$\tilde{J}_{0\pm}=\frac{3(-a_1\mp\sqrt{a_1^2+16a_2q^2})}{4a_2}. \tag{20}$$

Consistent with the Cartesian sign rule applied to $R(J)$, the parameters of the algebraic solitons are restricted by

$$a_2<0, \quad a_1>0. \tag{21}$$

Evaluation of $0<\tilde{J}_{\infty\pm}<\tilde{J}_{0\pm}$ leads to necessary and sufficient conditions for the existence of bright algebraic solitons. In addition to condition (21), either

$$a_1^2+16a_2q^2\geqslant 0 \tag{22a}$$

and

$$a_1^2+\frac{64}{3}a_2q^2<0, \tag{22b}$$

where $\tilde{J}_{\infty+}$ or $\tilde{J}_{\infty-}$ can be taken for the evaluation of $J(z), g(z), P$, or

$$a_1^2+\frac{64}{3}a_2q^2\geqslant 0, \tag{23}$$

where only $\tilde{J}_{\infty+}$ must be taken for the evaluation of Eqs. (5), (7), and (1). As the evaluation of $0<\tilde{J}_{0\pm}<\tilde{J}_{\infty\pm}$ shows, necessary and sufficient conditions for the existence of dark algebraic solitons are condition (21) and

$$a_1^2+\frac{64}{3}a_2q^2>0, \tag{24}$$

where only $\tilde{J}_{\infty-}$ must be chosen for the evaluation of $J(z), g(z), P$.

Thus, inserting the appropriate $\tilde{J}_{\infty\pm}, \tilde{J}_{0\pm}$ given by Eqs. (19) and (20) into Eq. (5), the intensity $J(z)$ of all bright and dark algebraic solitons can be evaluated explicitly, leading to

$$J_{\pm}(z)=\frac{36a_2a_1-a_1^3z^2\pm\sqrt{a_1^2+16a_2q^2}[(a_1^2+64a_2q^2)z^2+36a_2]}{4a_2[a_1(5a_1\pm 4\sqrt{a_1^2+16a_2q^2})z^2+4a_2(16q^2z^2-3)]}. \tag{25}$$

Subject to conditions (22) and (23), $J_+(z)$ and $J_-(z)$ represent a bright soliton (only one is stable). If condition (24) holds, $J_-(z)$ represents a dark soliton.

According to Eqs. (7) and (1), the phase function $g(z)$ and the normalized momentum P for algebraic solitons are determined by

$$g=\tilde{g}_{\pm}(z)=q\left(z-\frac{12\sqrt{6}R'\tanh^{-1}\left(\dfrac{R''\sqrt{\tilde{J}_{\infty\pm}}z}{2\sqrt{6}\sqrt{R''\tilde{J}_{\infty\pm}+6R'}}\right)}{R''\sqrt{\tilde{J}_{\infty\pm}(R''\tilde{J}_{\infty\pm}+6R')}}\Bigg|_{J=\tilde{J}_{0\pm}}\right), \tag{26}$$

$$P=\tilde{P}_{\pm}=-\frac{3\sqrt{6}\pi qR'}{R''}\left(\sqrt{-\frac{1}{R''}}+\sqrt{-\frac{\tilde{J}_{\infty\pm}}{6R'+R''\tilde{J}_{\infty\pm}}}\right)\Bigg|_{J=\tilde{J}_{0\pm}}, \tag{27}$$

with

$$R=\tilde{R}_{\pm}(J)=4\left(\frac{a_2}{3}J^4+\frac{a_1}{2}J^3-\lambda_{\pm}J^2+k_{\pm}J-q^2\tilde{J}_{\infty\pm}^2\right). \tag{28}$$

The signs in Eqs. (26) and (27) are associated consistent with conditions (22)–(24).

V. EXAMPLE AND SUMMARY

An example can elucidate the foregoing procedure. Selecting $a_1=3$, and $a_2=-1$, all stable solitons of the NLSE (3), given by Eq. (2), can be determined by finding the appropriate parameters q,λ. Condition $g_3(q,\lambda)\leqslant 0$ [or, equivalently, condition (14b)] defines a subset of the (q-λ) plane (see Fig. 1), for which condition (14a) is fulfilled. Points q,λ on the boundary C of this subset are associated to algebraic solitons, since $g_3(q,\lambda)=0$ on C. Points inside C belong to nonalgebraic solitons. Outside C, real solitons of the NLSE do not exist. Evaluation of Eqs. (5) and (15) with $J_\infty=J_{\infty+}$ and J_{0-}, J_{0+} yields bright and dark nonalgebraic solitons, respectively, as shown in Fig. 2. Since $\partial P_+/\partial q>0, \partial P_-/\partial q<0$, only dark nonalgebraic solitons are stable (see Fig. 1).

For algebraic solitons, the upper and the lower branch of C are represented by λ_- and λ_+ [according to Eqs. (18) and (19)], respectively. Condition (24) is fulfilled for $|q|<0.649$ so that dark solitons $J_-(z)$ exist according to Eq. (25). Since $\partial\tilde{P}_-/\partial q>0$ for $|q|<0.649$, algebraic dark solitons are stable. Subject to condition (23), the bright algebraic solitons are unstable since $\partial\tilde{P}_+/\partial q<0$ for $|q|<0.649$. Conditions (22a) and (22b) are fulfilled for $0.649<|q|<0.750$ so that bright solitons $J_+(z)$ and $J_-(z)$ exist. As shown in Fig. 1, only $\partial\tilde{P}_+/\partial q$ is positive for q subject to conditions (22a) and (22b), leading to stable $J_+(z)$ and unstable $J_-(z)$. As depicted in Fig. 3, the numerical simulation yields a transition from a stable dark soliton $J_-(z)$ to an unstable bright soliton at $q=0.649$. All results summarized in Fig. 1 depend on a_1, a_2. For arbitrary a_1, a_2, the discussion of the existence

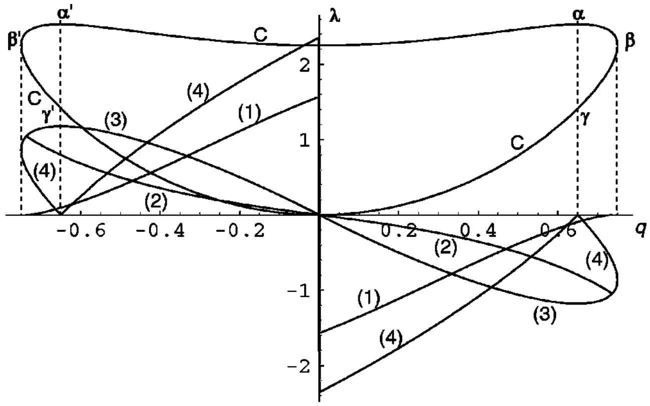

FIG. 1. Region of existence $g_3(q,\lambda)\leqslant 0$ (bounded by the closed curve C) and dependence of the normalized momentum P on parameters q and λ for $a_1=3$, $a_2=-1$. Inside C, nonalgebraic solitons. On C, algebraic solitons: $\alpha\alpha'$ stable dark; $\alpha\gamma$ and $\beta'\gamma'$ stable bright; $\gamma\gamma'$ unstable bright; curves (1), (2), (3), (4) denote $P_+, P_-, \tilde{P}_+, \tilde{P}_-$, respectively.

and stability criteria is intricate and requires further investigation.

To sum up, we have presented, to our knowledge for the first time, an analysis that contains analytical existence criteria for solitons of the NLSE (3) and exact analytical expressions for the intensity, phase, and normalized momentum.

In particular, necessary and sufficient for the existence of nonalgebraic ($g_3<0$) bright and dark solitons are, if $a_2<0$, the conditions (13a), (14a), (14b), and $a_1>0$. If $a_2>0$, in addition to conditions (13a) and (13b), condition (13c) must be valid.

Since the normalized momentum $P_\pm(a_1,a_2,\lambda,q)$ is given analytically by Eq. (16), the stability criterion $(\partial P_\pm/\partial q>0)$ represents the general dependence of stability with respect to the parameters a_1,a_2,λ and the soliton velocity $v=-q$.

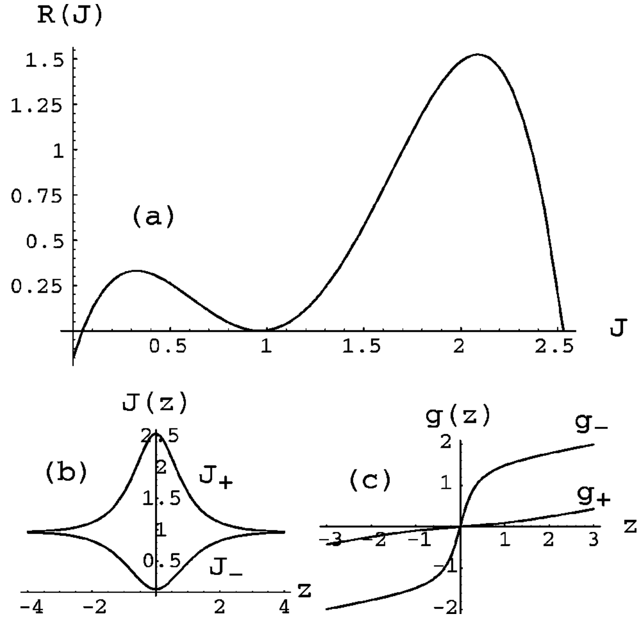

FIG. 2. Phase diagram (a) and profiles of $J(z)$ [(b)] and $g(z)$ [(c)] of nonalgebraic solitons (for $a_1=3,a_2=1,\lambda=2,q=0.2$).

Necessary and sufficient for the existence of algebraic solitons ($g_3=0$) are $a_2<0$, $a_1>0$, and, for dark solitons, condition (24). Bright algebraic solitions exist, if, in addition to $a_2<0,a_1>0$, conditions (22a) and (22b) or [instead of conditions (22a) and (22b)] condition (23) is satisfied. Since λ is related to the parameters a_1,a_2,q by Eq. (18), in this case the normalized momentum $\tilde{P}_\pm$ depends on the parameters a_1,a_2,q only. Thus, the parameter dependence of the stability criterion is simplified to a certain extent.

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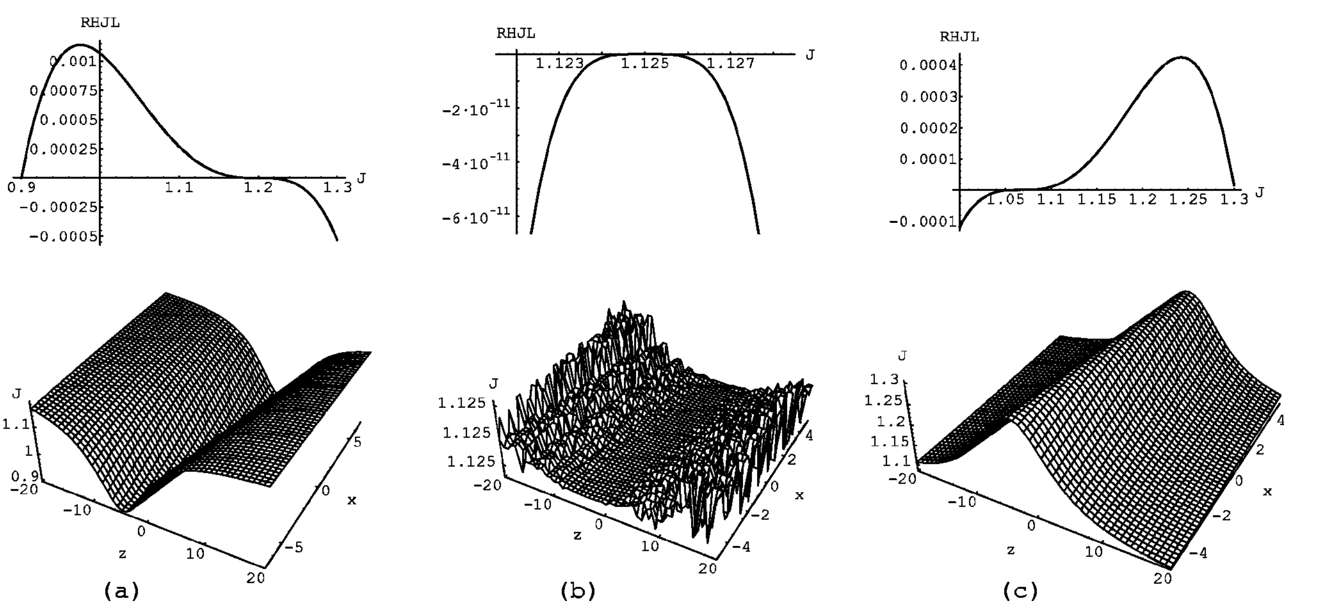

FIG. 3. Phase diagrams and intensity profiles of algebraic solitons: (a) at $q=0.600$, $\lambda=2.519$; (b) at $q=0.649$, $\lambda=2.531$; (c) at $q=0.680$, $\lambda=2.524$.

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